

Determinant representations for scalar products of the XXZ Gaudin model with general boundary terms

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Abstract

We obtain the determinant representations of the scalar products for the XXZ Gaudin model with generic non-diagonal boundary terms.

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1 Introduction

Gaudin type models [1] have important applications in physics. For instance, the XXZ Gaudin model has played an essential role in the study of the reduced BCS model whose exact solutions were first found by Richardson [2]. In fact, the conserved operators for the BCS model can be mapped to a set of XXX Gaudin Hamiltonians in a non-uniform magnetic field and the BCS Hamiltonian is expressible in terms of these operators [3, 4, 5]. This result also exposes a relationship between the BCS Hamiltonian and the perturbed WZNW model at critical level [6] based on the connection of Gaudin models with the solutions of the KZ equations [7, 8].

Eigenstates and the corresponding eigenvalues for the open XXZ Gaudin model for the boundary conditions with three free boundary parameters were derived in Ref.[9]. In this paper, we consider the most generic boundary conditions specified by the non-diagonal K-matrices in [10, 11], leading to the open XXZ Gaudin model in this paper which depends on four free boundary parameters. We compute the scalar products of this Gaudin model, and give their explicit expressions in terms of determinants, that is the determinant representations of the scalar products.

This paper is organized as follows. Section 2 provides some preliminaries on the boundary inverse scattering method. In section 3, we briefly describe the open XXZ Gaudin magnet associated with non-diagonal boundary K-matrices. In section 4, we derive the Bethe ansatz equations for the open XXZ Gaudin model and the symmetric, polarization-free expressions for the pseudo-particle creation operators. In section 5, we obtain the determinant representations for the partition and correlation functions of the model. We summarize our results in section 6 and present the details of some derivations and proofs in the Appendices.

2 Preliminaries: the inhomogeneous spin- $\frac{1}{2}$ XXZ open chain

Let V be a two-dimensional linear space and σ^\pm, σ^z be the Pauli matrices which give the spin- $\frac{1}{2}$ representation of $su(2)$ on V . The spin- $\frac{1}{2}$ XXZ chain can be constructed from the

well-known six-vertex model R-matrix $R(u) \in \text{End}(V \otimes V)$ [12] given by

$$R(u) = \begin{pmatrix} 1 & & \\ \frac{\sin u}{\sin(u+\eta)} & \frac{\sin \eta}{\sin(u+\eta)} & \\ \frac{\sin \eta}{\sin(u+\eta)} & \frac{\sin u}{\sin(u+\eta)} & \\ & & 1 \end{pmatrix}. \quad (2.1)$$

Here we assume η is a generic complex number. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE),

$$R_{1,2}(u_1 - u_2)R_{1,3}(u_1 - u_3)R_{2,3}(u_2 - u_3) = R_{2,3}(u_2 - u_3)R_{1,3}(u_1 - u_3)R_{1,2}(u_1 - u_2), \quad (2.2)$$

and the unitarity, crossing-unitarity and quasi-classical properties [9]. We adopt the standard notations: for any matrix $A \in \text{End}(V)$, A_j is an embedding operator in the tensor space $V \otimes V \otimes \cdots$, which acts as A on the j -th space and as identity on the other factor spaces; $R_{i,j}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as identity on the factor spaces except for the i -th and j -th ones.

One introduces the “row-to-row” (or one-row) monodromy matrix $T(u)$, which is an 2×2 matrix with elements being operators acting on $V^{\otimes N}$, where $N = 2M$ (M being a positive integer),

$$T_0(u) = R_{0,N}(u - z_N)R_{0,N-1}(u - z_{N-1}) \cdots R_{0,1}(u - z_1). \quad (2.3)$$

Here $\{z_j | j = 1, \dots, N\}$ are arbitrary free complex parameters which are usually called inhomogeneous parameters.

Integrable open chain can be constructed as follows [13]. Let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE)

$$\begin{aligned} R_{1,2}(u_1 - u_2)K_1^-(u_1)R_{2,1}(u_1 + u_2)K_2^-(u_2) \\ = K_2^-(u_2)R_{1,2}(u_1 + u_2)K_1^-(u_1)R_{2,1}(u_1 - u_2), \end{aligned} \quad (2.4)$$

and the latter satisfies the dual RE

$$\begin{aligned} R_{1,2}(u_2 - u_1)K_1^+(u_1)R_{2,1}(-u_1 - u_2 - 2\eta)K_2^+(u_2) \\ = K_2^+(u_2)R_{1,2}(-u_1 - u_2 - 2\eta)K_1^+(u_1)R_{2,1}(u_2 - u_1). \end{aligned} \quad (2.5)$$

For open spin-chains, instead of the standard “row-to-row” monodromy matrix $T(u)$ (2.3), one needs to consider the “double-row” monodromy matrix $\mathbb{T}(u)$

$$\mathbb{T}(u) = T(u)K^-(u)\hat{T}(u), \quad \hat{T}(u) = T^{-1}(-u). \quad (2.6)$$

Then the double-row transfer matrix of the XXZ chain with open boundary (or the open XXZ chain) is given by

$$\tau(u) = \text{tr}(K^+(u)\mathbb{T}(u)). \quad (2.7)$$

The QYBE and (dual) REs lead to that the transfer matrices with different spectral parameters commute with each other [13]: $[\tau(u), \tau(v)] = 0$. This ensures the integrability of the open XXZ chain.

3 XXZ Gaudin model with generic boundaries

We will consider the K-matrix $K^-(u)$ which is a generic solution to the RE (2.4) associated the six-vertex model R-matrix [10, 11]

$$K^-(u) = \begin{pmatrix} k_1^1(u) & k_2^1(u) \\ k_1^2(u) & k_2^2(u) \end{pmatrix} \equiv K(u). \quad (3.1)$$

The matrix elements are

$$\begin{aligned} k_1^1(u) &= \frac{\cos(\lambda_1 - \lambda_2) - \cos(\lambda_1 + \lambda_2 + 2\xi)e^{-2iu}}{2 \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u)}, \\ k_2^1(u) &= \frac{-i \sin(2u)e^{-i(\lambda_1 + \lambda_2)}e^{-iu}}{2 \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u)}, \\ k_1^2(u) &= \frac{i \sin(2u)e^{i(\lambda_1 + \lambda_2)}e^{-iu}}{2 \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u)}, \\ k_2^2(u) &= \frac{\cos(\lambda_1 - \lambda_2)e^{-2iu} - \cos(\lambda_1 + \lambda_2 + 2\xi)}{2 \sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u)}. \end{aligned} \quad (3.2)$$

The corresponding *dual* K-matrix $K^+(u)$ is a generic solution to the dual reflection equation (2.5) with a particular choice of the free boundary parameters:

$$K^+(u) = \begin{pmatrix} k^{+1}_1(u) & k^{+1}_2(u) \\ k^{+2}_1(u) & k^{+2}_2(u) \end{pmatrix} \quad (3.3)$$

with matrix elements

$$\begin{aligned} k^{+1}_1(u) &= \frac{\cos(\lambda_1 - \lambda_2)e^{-i\eta} - \cos(\lambda_1 + \lambda_2 + 2\bar{\xi})e^{2iu+i\eta}}{2 \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta)}, \\ k^{+1}_2(u) &= \frac{i \sin(2u + 2\eta)e^{-i(\lambda_1 + \lambda_2)}e^{iu-i\eta}}{2 \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta)}, \\ k^{+2}_1(u) &= \frac{-i \sin(2u + 2\eta)e^{i(\lambda_1 + \lambda_2)}e^{iu+i\eta}}{2 \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta)}, \\ k^{+2}_2(u) &= \frac{\cos(\lambda_1 - \lambda_2)e^{2iu+i\eta} - \cos(\lambda_1 + \lambda_2 + 2\bar{\xi})e^{-i\eta}}{2 \sin(\lambda_1 + \bar{\xi} - u - \eta) \sin(\lambda_2 + \bar{\xi} - u - \eta)}. \end{aligned} \quad (3.4)$$

The K-matrices depend on four free boundary parameters $\{\lambda_1, \lambda_2, \xi, \bar{\xi}\}$ which specify integrable boundary conditions [11]. We remark that $K^-(u)$ does not depend on the crossing parameter η but $K^+(u)$ does. The parameter $\bar{\xi}$ is required to have the following expansion:

$$\bar{\xi} = \xi + \eta\Delta + O(\eta^2), \quad \eta \rightarrow 0. \quad (3.5)$$

This results in the relation,

$$\lim_{\eta \rightarrow 0} \{K^+(u)K^-(u)\} = \lim_{\eta \rightarrow 0} \{K^+(u)\}K(u) = \text{id}. \quad (3.6)$$

Let us introduce the generalized XXZ Gaudin operators [1] $\{H_j | j = 1, 2, \dots, N\}$ associated with the spin- $\frac{1}{2}$ XXZ model with boundaries specified by the K-matrices (3.1) and (3.3):

$$\begin{aligned} H_j = & \Gamma_j(z_j) + \sum_{k \neq j}^{2M} \frac{1}{\sin(z_j - z_k)} \left\{ \sigma_k^+ \sigma_j^- + \sigma_k^- \sigma_j^+ + \cos(z_j - z_k) \frac{\sigma_k^z \sigma_j^z - 1}{2} \right\} \\ & + \sum_{k \neq j}^{2M} \frac{K_j^{-1}(z_j)}{\sin(z_j + z_k)} \left\{ \sigma_j^+ \sigma_k^- + \sigma_j^- \sigma_k^+ + \cos(z_j + z_k) \frac{\sigma_j^z \sigma_k^z - 1}{2} \right\} K_j(z_j), \end{aligned} \quad (3.7)$$

where $\Gamma_j(u) = \frac{\partial}{\partial \eta} \{\bar{K}_j(u)\}|_{\eta=0} K_j(u)$, $j = 1, \dots, N$, with $\bar{K}_j(u) = \text{tr}_0 \{K_0^+(u) R_{0j}(2u) P_{0j}\}$, and $\{z_j\}$ correspond to the inhomogeneous parameters of the spin- $\frac{1}{2}$ XXZ chain with generic open boundaries. For a generic choice of the boundary parameters $\{\lambda_1, \lambda_2, \xi, \bar{\xi}\}$, $\Gamma_j(u)$ is a non-diagonal matrix, in contrast to the situation in [3].

The XXZ Gaudin operators (3.7) can be obtained by expanding the double-row transfer matrix $\tau(u)$ (2.7) at $u = z_j$ around $\eta = 0$ [9]:

$$\tau(z_j) = \text{id} + \eta H_j + O(\eta^2), \quad j = 1, \dots, N, \quad (3.8)$$

$$H_j = \frac{\partial}{\partial \eta} \tau(z_j)|_{\eta=0}. \quad (3.9)$$

Then the commutativity of the transfer matrices $\{\tau(z_j)\}$ for a generic η implies that

$$[H_j, H_k] = 0, \quad i, j = 1, \dots, N. \quad (3.10)$$

Thus the Gaudin model with the local Hamiltonian (3.7) is integrable.

4 Eigenstates and the corresponding eigenvalues

The relation (3.9) between $\{H_j\}$ and $\{\tau(z_j)\}$ and the fact that the first term of (3.8) is the identity operator enable us to extract the eigenstates of the Gaudin operators and the

corresponding eigenvalues from those of the XXZ chain obtained in [14, 15, 9, 16, 17, 18, 19, 20, 21, 22, 23].

Let us introduce the states $|\Omega^{(1)}\rangle$ and $|\Omega^{(2)}\rangle$,

$$|\Omega^{(1)}\rangle = \begin{pmatrix} e^{-i(z_1+2\lambda_1)} \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} e^{-i(z_N+2\lambda_1)} \\ 1 \end{pmatrix}, \quad (4.1)$$

$$|\Omega^{(2)}\rangle = \begin{pmatrix} e^{-i(z_1+2\lambda_2)} \\ 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} e^{-i(z_N+2\lambda_2)} \\ 1 \end{pmatrix}, \quad (4.2)$$

and the matrix $g(u) \in \text{End}(V)$ and the associated gauged Pauli operator $\sigma^\pm(u) \in \text{End}(V)$

$$g(u) = \begin{pmatrix} e^{-i(u+2\lambda_1)} & e^{-i(u+2\lambda_2)} \\ 1 & 1 \end{pmatrix}, \quad (4.3)$$

$$\sigma^\pm(u) = g(u)\sigma^\pm g(u)^{-1}. \quad (4.4)$$

Then we define the states,

$$|\{v_i^{(1)}\}\rangle^{(1)} = \prod_{i=1}^M B(v_i^{(1)})|\Omega^{(1)}\rangle, \quad (4.5)$$

$$|\{v_i^{(2)}\}\rangle^{(2)} = \prod_{i=1}^M C(v_i^{(2)})|\Omega^{(2)}\rangle. \quad (4.6)$$

The associated operators $B(u)$ and $C(u)$ are

$$B(u) = \sum_{i=1}^N \frac{\sin(\lambda_1 + \xi + z_i) \sin(\lambda_2 + \xi - z_i) \cdot \sin(2u)}{\sin(\lambda_1 + \xi - u) \sin(\lambda_2 + \xi - u) \sin(u - z_i) \sin(u + z_i)} \times \sigma^-(z_i), \quad (4.7)$$

$$C(u) = \sum_{i=1}^N \frac{\sin(\lambda_1 + \xi - z_i) \sin(\lambda_2 + \xi + z_i) \cdot \sin(2u)}{\sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u) \sin(u - z_i) \sin(u + z_i)} \times \sigma^+(z_i). \quad (4.8)$$

Using the same method as in [24], we can show that the above states (4.5) and (4.6) are the common eigenstates of the Gaudin operators $\{H_j\}$ given by (3.7) provided that the parameters $\{v_i^{(i)}\}$ satisfy the following two sets of Bethe ansatz equations

$$\begin{aligned} & \frac{1 - \Delta}{\sin(\lambda_1 + \xi + v_\alpha^{(1)}) \sin(\lambda_1 + \xi - v_\alpha^{(1)})} + \frac{1 + \Delta}{\sin(\lambda_2 + \xi + v_\alpha^{(1)}) \sin(\lambda_2 + \xi - v_\alpha^{(1)})} \\ = & \sum_{k \neq \alpha}^M \frac{2}{\sin(v_\alpha^{(1)} - v_k^{(1)}) \sin(v_\alpha^{(1)} + v_k^{(1)})} - \sum_{k=1}^{2M} \frac{1}{\sin(v_\alpha^{(1)} - z_k) \sin(v_\alpha^{(1)} + z_k)}, \end{aligned} \quad (4.9)$$

$$\begin{aligned} & \frac{1 + \Delta}{\sin(\lambda_1 + \xi + v_\alpha^{(2)}) \sin(\lambda_1 + \xi - v_\alpha^{(2)})} + \frac{1 - \Delta}{\sin(\lambda_2 + \xi + v_\alpha^{(2)}) \sin(\lambda_2 + \xi - v_\alpha^{(2)})} \\ = & \sum_{k \neq \alpha}^M \frac{2}{\sin(v_\alpha^{(2)} - v_k^{(2)}) \sin(v_\alpha^{(2)} + v_k^{(2)})} - \sum_{k=1}^{2M} \frac{1}{\sin(v_\alpha^{(2)} - z_k) \sin(v_\alpha^{(2)} + z_k)}, \\ & \alpha = 1, \dots, M. \end{aligned} \quad (4.10)$$

Here Δ is the parameter of first order expansion of $\bar{\xi}$ in terms of η , as defined in (3.5). Namely,

$$H_j |\{v_\alpha^{(i)}\}\rangle^{(i)} = E_j^{(i)} |\{v_\alpha^{(i)}\}\rangle^{(i)}, \quad i = 1, 2, \quad (4.11)$$

where $E_j^{(i)}$ are given by

$$\begin{aligned} E_j^{(1)} &= \cot 2z_j + \sum_{j=1}^2 \cot(\lambda_j + \xi - z_j) - \frac{\Delta \sin(2z_j)}{\sin(\lambda_1 + \xi - z_j) \sin(\lambda_1 + \xi + z_j)} \\ &\quad + \sum_{k=1}^M \frac{\sin(2z_j)}{\sin(v_k^{(1)} - z_j) \sin(v_k^{(1)} + z_j)}, \end{aligned} \quad (4.12)$$

$$\begin{aligned} E_j^{(2)} &= \cot 2z_j + \sum_{j=1}^2 \cot(\lambda_j + \xi - z_j) - \frac{\Delta \sin(2z_j)}{\sin(\lambda_2 + \xi - z_j) \sin(\lambda_2 + \xi + z_j)} \\ &\quad + \sum_{k=1}^M \frac{\sin(2z_j)}{\sin(v_k^{(2)} - z_j) \sin(v_k^{(2)} + z_j)}. \end{aligned} \quad (4.13)$$

5 Determinant representations of the scalar products

To obtain correlation functions, it suffices to calculate the scalar products of on-shell Bethe states with general off-shell Bethe states [12] (see also [25, 26] for the open XXZ chain with diagonal boundaries). In this section, we will obtain the explicit expressions of the following scalar products for the open XXZ Gaudin model with non-diagonal boundary terms:

$$S^{1,2}(\{u_\alpha\}; \{v_i^{(2)}\}) = {}^{(1)}\langle \{u_\alpha\} | \{v_i^{(2)}\} \rangle^{(2)}, \quad S^{2,1}(\{u_\alpha\}; \{v_i^{(1)}\}) = {}^{(2)}\langle \{u_\alpha\} | \{v_i^{(1)}\} \rangle^{(1)}, \quad (5.1)$$

$$S^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\}) = {}^{(1)}\langle \{u_\alpha\} | \{v_i^{(1)}\} \rangle^{(1)}, \quad S^{2,2}(\{u_\alpha\}; \{v_i^{(2)}\}) = {}^{(2)}\langle \{u_\alpha\} | \{v_i^{(2)}\} \rangle^{(2)}, \quad (5.2)$$

where ${}^{(1)}\langle \{u_\alpha\} |$ and ${}^{(2)}\langle \{u_\alpha\} |$ are defined by

$${}^{(1)}\langle \{u_\alpha\} | = \langle \Omega^{(1)} | C(u_M) \dots C(u_1), \quad (5.3)$$

$${}^{(2)}\langle \{u_\alpha\} | = \langle \Omega^{(2)} | B(u_M) \dots B(u_1), \quad (5.4)$$

with $\langle \Omega^{(1)} |$, $\langle \Omega^{(2)} |$ being the dual states of $|\Omega^{(1)}\rangle$, $|\Omega^{(2)}\rangle$, respectively,

$$\langle \Omega^{(1)} | = \left\{ \prod_{j=1}^N \frac{ie^{-i(z_j + \lambda_1 + \lambda_2)}}{2 \sin(\lambda_1 - \lambda_2)} \right\} (1, -e^{-i(z_1 + 2\lambda_2)}) \otimes \dots \otimes (1, -e^{-i(z_N + 2\lambda_2)}), \quad (5.5)$$

$$\langle \Omega^{(2)} | = \left\{ \prod_{j=1}^N \frac{ie^{-i(z_j + \lambda_1 + \lambda_2)}}{2 \sin(\lambda_1 - \lambda_2)} \right\} (-1, e^{-i(z_1 + 2\lambda_1)}) \otimes \dots \otimes (-1, e^{-i(z_N + 2\lambda_1)}). \quad (5.6)$$

By means of (4.1)-(4.8), (5.5) and (5.6), the scalar products can be written as

$$S^{1,2}(\{u_\alpha\}; \{v_i\}) = \langle \uparrow | \tilde{C}(u_1) \cdots \tilde{C}(u_M) \tilde{C}(v_1) \cdots \tilde{C}(v_M) | \downarrow \rangle, \quad (5.7)$$

$$S^{2,1}(\{u_\alpha\}; \{v_i\}) = \langle \downarrow | \tilde{B}(u_1) \cdots \tilde{B}(u_M) \tilde{B}(v_1) \cdots \tilde{B}(v_M) | \uparrow \rangle, \quad (5.8)$$

$$S^{1,1}(\{u_\alpha\}; \{v_i\}) = \langle \uparrow | \tilde{C}(u_1) \cdots \tilde{C}(u_M) \tilde{B}(v_1) \cdots \tilde{B}(v_M) | \uparrow \rangle, \quad (5.9)$$

$$S^{2,2}(\{u_\alpha\}; \{v_i\}) = \langle \downarrow | \tilde{B}(u_1) \cdots \tilde{B}(u_M) \tilde{C}(v_1) \cdots \tilde{C}(v_M) | \downarrow \rangle. \quad (5.10)$$

Here $|\uparrow\rangle$ and $\langle\uparrow|$ (resp. $|\downarrow\rangle$ and $\langle\downarrow|$) are the all spin-up state and its dual (resp. all spin-down and its dual), and $\tilde{C}(u)$ and $\tilde{B}(u)$ are given by

$$\tilde{B}(u) = \sum_{i=1}^N \frac{\sin(\lambda_1 + \xi + z_i) \sin(\lambda_2 + \xi - z_i) \sin(2u)}{\sin(\lambda_1 + \xi - u) \sin(\lambda_2 + \xi - u) \sin(u - z_i) \sin(u + z_i)} \times \sigma_i^-, \quad (5.11)$$

$$\tilde{C}(u) = \sum_{i=1}^N \frac{\sin(\lambda_1 + \xi - z_i) \sin(\lambda_2 + \xi + z_i) \sin(2u)}{\sin(\lambda_1 + \xi + u) \sin(\lambda_2 + \xi + u) \sin(u - z_i) \sin(u + z_i)} \times \sigma_i^+. \quad (5.12)$$

5.1 Scalar products $S^{1,2}$ and $S^{2,1}$

Let us introduce two functions

$$Z_N^{(1)}(\{\bar{u}_J\}) \equiv S^{1,2}(\{u_\alpha\}; \{v_i\}) = \langle \uparrow | \tilde{C}(u_1) \cdots \tilde{C}(u_M) \tilde{C}(v_1) \cdots \tilde{C}(v_M) | \downarrow \rangle, \quad (5.13)$$

$$Z_N^{(2)}(\{\bar{u}_J\}) \equiv S^{2,1}(\{u_\alpha\}; \{v_i\}) = \langle \downarrow | \tilde{B}(u_1) \cdots \tilde{B}(u_M) \tilde{B}(v_1) \cdots \tilde{B}(v_M) | \uparrow \rangle, \quad (5.14)$$

where the N parameters $\{\bar{u}_J | J = 1, \dots, N\}$ are defined as

$$\bar{u}_i = u_i \text{ for } i = 1, \dots, M, \quad \text{and} \quad \bar{u}_{M+i} = v_i \text{ for } i = 1, \dots, M. \quad (5.15)$$

Note that these functions $Z_N^{(1)}(\{\bar{u}_J\})$ and $Z_N^{(2)}(\{\bar{u}_J\})$ may correspond to the partition functions of the Gaudin model with domain wall boundary condition and one reflecting end [27, 28] specified by the non-diagonal K-matrices (3.1) and (3.3).

We find that the functions $Z_N^{(k)}(\{\bar{u}_J\})$ above can be expressed in term of the determinants of the $N \times N$ matrices $\mathcal{N}^{(k)}(\{\bar{u}_\alpha\}; \{z_i\})_{\alpha,j}$,

$$Z_N^{(k)}(\{\bar{u}_J\}) = \frac{\prod_{\alpha=1}^N \prod_{i=1}^N \sin(\bar{u}_\alpha + z_i) \sin(\bar{u}_\alpha - z_i) \det \mathcal{N}^{(k)}(\{\bar{u}_\alpha\}; \{z_i\})_{\alpha,j}}{\prod_{\alpha>\beta} \sin(\bar{u}_\alpha - \bar{u}_\beta) \sin(\bar{u}_\alpha + \bar{u}_\beta) \prod_{k<j} \sin(z_k - z_j) \sin(z_k + z_j)}, \quad (5.16)$$

where

$$\mathcal{N}^{(1)}(\{\bar{u}_\alpha\}; \{z_i\})_{\alpha,j} = \frac{\sin(\lambda_1 + \xi - z_i) \sin(\lambda_2 + \xi + z_i) \sin(2\bar{u})}{\sin(\lambda_1 + \xi + \bar{u}) \sin(\lambda_2 + \xi + \bar{u}) \sin^2(\bar{u} - z_i) \sin^2(\bar{u} + z_i)}, \quad (5.17)$$

$$\mathcal{N}^{(2)}(\{\bar{u}_\alpha\}; \{z_i\})_{\alpha,j} = \frac{\sin(\lambda_1 + \xi + z_i) \sin(\lambda_2 + \xi - z_i) \sin(2\bar{u})}{\sin(\lambda_1 + \xi - \bar{u}) \sin(\lambda_2 + \xi - \bar{u}) \sin^2(\bar{u} - z_i) \sin^2(\bar{u} + z_i)}. \quad (5.18)$$

In appendix A we give the proof of this determinant representation of the partition functions.

5.2 Scalar products $S^{1,1}$ and $S^{2,2}$

In this subsection, we calculate the scalar products $S^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\})$ (5.9) and $S^{2,2}(\{u_\alpha\}; \{v_i^{(2)}\})$ (5.10). In this case, we assume that $\{v_i^{(k)}\}$ satisfy the associated Bethe ansatz equations.

Let us first consider the scalar product $S^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\})$ (5.9). We insert in the scalar product a sum over the complete set of states $|j_1, \dots, j_M\rangle$ between each operator, where $|j_1, \dots, j_i\rangle$ is the state with i spins down at the sites j_1, \dots, j_i and $2M - i$ spins up at the other sites. We are thus led to considering the intermediate functions

$$G^{(i)}(u_1, \dots, u_i | j_{i+1}, \dots, j_M; \{v_i^{(1)}\}) = \langle j_{i+1}, \dots, j_M | \tilde{C}(u_i) \cdots \tilde{C}(u_1) \times \tilde{B}(v_1^{(1)}) \cdots \tilde{B}(v_M^{(1)}) | \uparrow \rangle, \quad (5.19)$$

which satisfy the following recursive relations:

$$\begin{aligned} & G^{(i)}(u_1, \dots, u_i | j_{i+1}, \dots, j_M; \{v_i^{(1)}\}) \\ &= \sum_{j \neq j_{i+1}, \dots, j_M} \langle j_{i+1}, \dots, j_M | \tilde{C}(u_i) | j, j_{i+1}, \dots, j_M \rangle \\ & \quad \times G^{(i-1)}(u_1, \dots, u_{i-1} | j, j_{i+1}, \dots, j_M; \{v_i^{(1)}\}), \quad i = 1, \dots, M. \end{aligned} \quad (5.20)$$

The last one of these functions is the scalar product $S^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\})$, namely,

$$G^{(M)}(u_1, \dots, u_M; \{v_i^{(1)}\}) = S^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\}), \quad (5.21)$$

whereas the first one,

$$G^{(0)}(j_1, \dots, j_M; \{v_i^{(1)}\}) = \langle j_1, \dots, j_M | \tilde{B}(v_1^{(1)}) \cdots \tilde{B}(v_M^{(1)}) | \uparrow \rangle \quad (5.22)$$

is closely related to the partition function $Z_N^{(2)}$ (5.16) of the Gaudin model.

We then perform the summation in (5.19) and compute successively the functions $G^{(i)}$. Details are given in Appendix B. Finally, the scalar product $S^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\})$ has the following determinant representation

$$\begin{aligned} & S_M^{1,1}(\{u_\alpha\}, \{v_i^{(1)}\}) = G^{(M)}(u_1, \dots, u_M; \{v_i^{(1)}\}) \\ &= \frac{\det \widetilde{\mathcal{N}}^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\})}{\prod_{k < j} \sin(u_k - u_j) \sin(u_k + u_j) \prod_{\alpha > \beta} \sin(v_\alpha^{(1)} - v_\beta^{(1)}) \sin(v_\alpha^{(1)} + v_\beta^{(1)})}, \end{aligned} \quad (5.23)$$

where the matrix $\widetilde{\mathcal{N}}^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\})$ is given by

$$\widetilde{\mathcal{N}}^{1,1}(\{u_\alpha\}; \{v_i^{(1)}\})_{\alpha,j} = \frac{\sin(2v_j^{(1)}) \sin(2u_\alpha) F_j^{(1)}(u_\alpha; \{z_i\}, \{v_i^{(1)}\})}{\sin(\lambda_2 + \xi - v_j^{(1)}) \sin(\lambda_1 + \xi - v_j^{(1)})}. \quad (5.24)$$

Here we have introduced the functions

$$\begin{aligned}
& F_j^{(1)}(u_\alpha; \{z_i\}, \{v_i^{(1)}\}) \\
&= \frac{\sin(\lambda_1 + \xi - u_\alpha) \sin(\lambda_2 + \xi - u_\alpha) \prod_{k=1}^M \sin(v_k^{(1)} - u_\alpha) \prod_{k=1}^M \sin(v_k^{(1)} + u_\alpha)}{\sin^2(v_j^{(1)} - u_\alpha) \sin^2(v_j^{(1)} + u_\alpha)} \\
& \quad \left[\frac{1 - \Delta}{\sin(\lambda_1 + \xi + u_\alpha) \sin(\lambda_1 + \xi - u_\alpha)} + \frac{1 + \Delta}{\sin(\lambda_2 + \xi + u_\alpha) \sin(\lambda_2 + \xi - u_\alpha)} \right. \\
& \quad \left. + \sum_{k=1}^{2M} \frac{1}{\sin(u_\alpha - z_k) \sin(u_\alpha + z_k)} - \sum_{k \neq \alpha}^M \frac{2}{\sin(u_\alpha - v_k^{(1)}) \sin(u_\alpha + v_k^{(1)})} \right]. \quad (5.25)
\end{aligned}$$

Similarly, we obtain the scalar product $S^{2,2}(\{u_\alpha\}; \{v_i^{(2)}\})$ in terms of determinants with the parameters $\{v_k^{(2)}\}$ satisfying the second set of Bethe ansatz equations (4.10),

$$\begin{aligned}
& S^{2,2}(\{u_\alpha\}; \{v_i^{(2)}\}) \\
&= \frac{\det \widetilde{\mathcal{N}}^{2,2}(\{u_\alpha\}; \{v_i^{(2)}\})}{\prod_{k < j} \sin(u_k - u_j) \sin(u_k + u_j) \prod_{\alpha > \beta} \sin(v_\alpha^{(2)} - v_\beta^{(2)}) \sin(v_\alpha^{(2)} + v_\beta^{(2)})}, \quad (5.26)
\end{aligned}$$

where the $M \times M$ matrix $\widetilde{\mathcal{N}}^{2,2}(\{u_\alpha\}; \{v_i^{(2)}\})$ is given by

$$\widetilde{\mathcal{N}}^{2,2}(\{u_\alpha\}; \{v_i^{(2)}\})_{\alpha,j} = \frac{\sin(2v_j^{(2)}) \sin(2u_\alpha) F_j^{(2)}(u_\alpha; \{z_i\}, \{v_i^{(2)}\})}{\sin(\lambda_2 + \xi + v_j^{(2)}) \sin(\lambda_1 + \xi + v_j^{(2)})}. \quad (5.27)$$

Here

$$\begin{aligned}
& F_j^{(2)}(u_\alpha; \{z_i\}, \{v_i^{(2)}\}) \\
&= \frac{\sin(\lambda_1 + \xi + u_\alpha) \sin(\lambda_2 + \xi + u_\alpha) \prod_{k=1}^M \sin(v_k^{(2)} - u_\alpha) \prod_{k=1}^M \sin(v_k^{(2)} + u_\alpha)}{\sin^2(v_j^{(2)} - u_\alpha) \sin^2(v_j^{(2)} + u_\alpha)} \\
& \quad \left[\frac{1 + \Delta}{\sin(\lambda_1 + \xi + u_\alpha) \sin(\lambda_1 + \xi - u_\alpha)} + \frac{1 - \Delta}{\sin(\lambda_2 + \xi + u_\alpha) \sin(\lambda_2 + \xi - u_\alpha)} \right. \\
& \quad \left. + \sum_{k=1}^{2M} \frac{1}{\sin(u_\alpha - z_k) \sin(u_\alpha + z_k)} - \sum_{k \neq \alpha}^M \frac{2}{\sin(u_\alpha - v_k^{(2)}) \sin(u_\alpha + v_k^{(2)})} \right]. \quad (5.28)
\end{aligned}$$

6 Conclusions

We have studied the XXZ Gaudin model with generic non-diagonal boundary terms. In addition to the inhomogeneous parameters $\{z_j\}$, the associated Gaudin operators $\{H_j\}$,

(3.7), depend on four free parameters $\{\lambda_1, \lambda_2, \xi, \Delta\}$. Thus our Gaudin operators are four-parameter $(\{\lambda_1, \lambda_2, \xi, \Delta\})$ generalizations of those in [29], three-parameter $(\{\lambda_1, \lambda_2, \xi\})$ generalizations of those in [24, 3], and one-parameter $(\{\Delta\})$ generalizations of those in [9]. The common eigenstates (Bethe states) of the operators are constructed by algebraic Bethe ansatz method. We have obtained the determinant representations (5.16), (5.23) and (5.28) of the scalar products for the boundary XXZ Gaudin model.

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Appendix A: Derivation of (5.16)

Take the partition function $Z_N^{(2)}(\{v_\alpha\}; \{z_i\})$ as an example, which satisfies the recursive relation,

$$\begin{aligned} Z_N^{(2)}(\{v_\alpha\}; \{z_i\}) &= \langle j_1, \dots, j_N | \tilde{B}(v_N) | j_1, \dots, j_{N-1} \rangle \\ &\quad \times \langle j_1, \dots, j_{N-1} | \tilde{B}(v_{N-1}) \cdots \tilde{B}(v_1) | \uparrow \rangle, \end{aligned} \quad (\text{A.1})$$

namely

$$\begin{aligned} Z_N^{(2)}(\{v_\alpha\}; \{z_i\}) &= \sum_{i=1}^N \frac{\sin(\lambda_1 + \xi + z_i) \sin(\lambda_2 + \xi - z_i) \sin(2v_N)}{\sin(\lambda_1 + \xi - v_N) \sin(\lambda_2 + \xi - v_N) \sin(v_N - z_i) \sin(v_N + z_i)} \\ &\quad \times Z_{N-1}^{(2)}(\{v_\alpha\}_{\alpha \neq N}; \{z_j\}_{j \neq i}). \end{aligned} \quad (\text{A.2})$$

The partition function $Z_N^{(2)}(\{v_\alpha\}; \{z_i\})$, for any positive integer N , can be uniquely determined by the initial condition: $Z_0^{(2)}(\{v_\alpha\}; \{z_i\}) = 1$ and the recursive relation (A.2).

Let us introduce a series of functions $\{K_I(\{v_\alpha\}; \{z_i\}) | I = 1, \dots, N\}$,

$$K_N(\{v_\alpha\}; \{z_i\})$$

$$\begin{aligned}
&= \frac{\prod_{\alpha=1}^N \prod_{i=1}^N \sin(v_\alpha + z_i) \sin(v_\alpha - z_i)}{\prod_{\alpha > \beta} \sin(v_\alpha - v_\beta) \sin(v_\alpha + v_\beta) \prod_{k < j} \sin(z_k - z_j) \sin(z_k + z_j)} \\
&\times \det \left| \frac{\sin(\lambda_1 + \xi + z_j) \sin(\lambda_2 + \xi - z_j) \cdot 2v_\alpha}{\sin(\lambda_1 + \xi - v_\alpha) \sin(\lambda_2 + \xi - v_\alpha) \sin^2(v_\alpha - z_j) \sin^2(v_\alpha + z_j)} \right|, \quad (\text{A.3})
\end{aligned}$$

and prove the relation,

$$Z_I^{(2)}(\{v_\alpha\}; \{z_i\}) = K_I(\{v_\alpha\}; \{z_i\}), \quad \text{for any positive integer } I. \quad (\text{A.4})$$

We use the induction method.

- We have, for the case of $N = 1$,

$$Z_1^{(2)}(v_1; z_1) = K_1(v_1; z_1) = \frac{\sin(\lambda_1 + \xi + z_1) \sin(\lambda_2 + \xi - z_1) \sin(2v_1)}{\sin(\lambda_1 + \xi - v_1) \sin(\lambda_2 + \xi - v_1) \sin(v_1 - z_1) \sin(v_1 + z_1)}. \quad (\text{A.5})$$

- Suppose that (A.4) holds for $I \leq N - 1$. We prove that (A.3) also holds for $I = N$ holds. This can be done as follows. The determinant representation of $K_N(\{v_\alpha\}; \{z_i\})$ implies that it satisfies the following recursive relation

$$\begin{aligned}
&K_N(\{v_\alpha\}; \{z_i\}) \\
&= \sum_{i=1}^N \frac{\sin(\lambda_1 + \xi + z_i) \sin(\lambda_2 + \xi - z_i) \sin(2v_N)}{\sin(\lambda_1 + \xi - v_N) \sin(\lambda_2 + \xi - v_N) \sin(v_N - z_i) \sin(v_N + z_i)} \\
&\times \prod_{l=1}^{N-1} \frac{\sin(v_l - z_i) \sin(v_l + z_i)}{\sin(v_N - v_l) \sin(v_N + v_l)} \prod_{j \neq i} \frac{\sin(v_N - z_j) \sin(v_N + z_j)}{\sin(z_j - z_i) \sin(z_j + z_i)} \\
&\times K_{N-1}(\{v_\alpha\}_{\alpha \neq N}; \{z_j\}_{j \neq i}). \quad (\text{A.6})
\end{aligned}$$

The determinant representation (A.3), the recursive relation (A.6) and the recursive relation (A.2) mean that $K_N(\{v_\alpha\}; \{z_i\})$ and $Z_N^{(2)}(\{v_\alpha\}; \{z_i\})$, as functions of v_N , have the same set of simple poles,

$$\pm z_i, \lambda_1 + \xi, \lambda_2 + \xi \pmod{2\pi}, \quad i = 1, \dots, N, \quad (\text{A.7})$$

at which both functions have the same residues. Moreover we can show that

$$Z_N^{(2)}(\{v_\alpha\}; \{z_i\})|_{v_N \rightarrow \infty} = 0 = K_N(\{v_\alpha\}; \{z_i\})|_{v_N \rightarrow \infty}. \quad (\text{A.8})$$

We thus conclude that (A.4) also holds for $I = N$. This completes the induction.

Finally we get the determinant representation of the partition function $Z_N^{(2)}(\{v_\alpha\}; \{z_i\})$,

$$Z_N^{(2)}(\{v_\alpha\}; \{z_i\}) = \frac{\prod_{\alpha=1}^N \prod_{i=1}^N \sin(v_\alpha + z_i) \sin(v_\alpha - z_i) \det \mathcal{N}^{(2)}}{\prod_{\alpha > \beta} \sin(v_\alpha - v_\beta) \sin(v_\alpha + v_\beta) \prod_{k < j} \sin(z_k - z_j) \sin(z_k + z_j)}, \quad (\text{A.9})$$

where the $N \times N$ matrix $\mathcal{N}^{(2)}(\{v_\alpha\}; \{z_j\})$ is given by

$$\mathcal{N}^{(2)}(\{v_\alpha\}; \{z_j\})_{\alpha,j} = \frac{\sin(\lambda_1 + \xi + z_j) \sin(\lambda_2 + \xi - z_j) \sin(2v_\alpha)}{\sin(\lambda_1 + \xi - v_\alpha) \sin(\lambda_2 + \xi - v_\alpha) \sin^2(v_\alpha - z_j) \sin^2(v_\alpha + z_j)}. \quad (\text{A.10})$$

Appendix B: Proof of (5.23)

We prove (5.23) by calculating the functions $G^{(i)}$ (5.19) recursively.

We illustrate our derivations for $G^{(1)}$. We firstly express $G^{(1)}$ in terms of $G^{(0)}$,

$$\begin{aligned} & G^{(1)}(u_1 | j_2, \dots, j_M; \{v_i^{(1)}\}) \\ &= \sum_{j \neq j_2, \dots, j_M} \langle j_2, \dots, j_M | \tilde{C}(u_1) | j, j_2, \dots, j_M \rangle \times G^{(0)}(j, j_2, \dots, j_M; \{v_i^{(1)}\}) \\ &= \sum_{j \neq j_2, \dots, j_M} \frac{\sin(\lambda_1 + \xi - z_i) \sin(\lambda_2 + \xi + z_i) \sin(2u_1)}{\sin(\lambda_1 + \xi + u_1) \sin(\lambda_2 + \xi + u_1) \sin(u_1 - z_i) \sin(u_1 + z_i)} \\ & \quad \times \frac{\prod_{\alpha=1}^M \prod_{i=1}^M \sin(v_\alpha^{(1)} + z_i) \sin(v_\alpha^{(1)} - z_i) \det \mathcal{N}(\{v_\alpha^{(1)}\}; \{z_i\})}{\prod_{\alpha > \beta} \sin(v_\alpha^{(1)} - v_\beta^{(1)}) \sin(v_\alpha^{(1)} + v_\beta^{(1)}) \prod_{k < j} \sin(z_k - z_j) \sin(z_k + z_j)}, \quad (\text{B.1}) \end{aligned}$$

where $\mathcal{N}(\{v_\alpha^{(1)}\}; \{z_i\})$ is a $M \times M$ matrix with matrix elements $\mathcal{N}(\{v_\alpha^{(1)}\}; \{z_i\})_{\alpha,j}$ given by

$$\mathcal{N}(\{v_\alpha^{(1)}\}; \{z_i\})_{\alpha,j} = \frac{\sin(\lambda_1 + \xi + z_i) \sin(\lambda_2 + \xi - z_i) \sin(2v_\alpha^{(1)})}{\sin(\lambda_1 + \xi - v_\alpha^{(1)}) \sin(\lambda_2 + \xi - v_\alpha^{(1)}) \sin^2(v_\alpha^{(1)} - z_i) \sin^2(v_\alpha^{(1)} + z_i)}. \quad (\text{B.2})$$

Further, performing the summation and absorbing the results into the element of the first column of the matrix $\mathcal{N}^{(1)}$ below, we have

$$\begin{aligned} & G^{(1)}(u_1 | j_2, \dots, j_M; \{v_i^{(1)}\}) \\ &= \frac{\prod_{\alpha=1}^M \prod_{k=2}^M \sin(v_\alpha^{(1)} + z_{i_k}) \sin(v_\alpha^{(1)} - z_{i_k}) \det \mathcal{N}^{(1)}(\{v_\alpha^{(1)}\}; u_1, i_2, \dots, i_m)}{\prod_{M \geq \alpha > \beta \geq 1} \sin(v_\alpha^{(1)} - v_\beta^{(1)}) \sin(v_\alpha^{(1)} + v_\beta^{(1)}) \prod_{2 \leq k < l \leq M} \sin(z_{i_k} - z_{i_l}) \sin(z_{i_k} + z_{i_l})}, \quad (\text{B.3}) \end{aligned}$$

where

$$\mathcal{N}_{ab}^{(1)} = \frac{1}{\sin(u_1 - z_{i_b}) \sin(u_1 + z_{i_b})} \mathcal{N}_{ab} \quad \text{for } b \geq 2, \quad (\text{B.4})$$

$$\begin{aligned}
\mathcal{N}_{a1}^{(1)} &= \prod_{k=2}^M \sin(u_1 - z_{i_k}) \sin(u_1 + z_{i_k}) \\
&\times \sum_{i_1 \neq i_2, \dots, i_M}^M \frac{\prod_{\alpha=1}^M \sin(v_\alpha^{(1)} + z_{i_1}) \sin(v_\alpha^{(1)} - z_{i_1})}{\prod_{k>1}^M \sin(z_{i_1} - z_{i_k}) \sin(z_{i_k} + z_{i_1})} \\
&\times \frac{\sin(\lambda_1 + \xi + z_{i_1}) \sin(\lambda_2 + \xi - z_{i_1}) \sin(2v_a^{(1)})}{\sin(\lambda_1 + \xi - v_a^{(1)}) \sin(\lambda_2 + \xi - v_a^{(1)}) \sin^2(v_a^{(1)} - z_{i_1}) \sin^2(v_a^{(1)} + z_{i_1})} \\
&\times \frac{\sin(\lambda_1 + \xi - z_{i_1}) \sin(\lambda_2 + \xi + z_{i_1}) \sin(2u_1)}{\sin(\lambda_1 + \xi + u_1) \sin(\lambda_2 + \xi + u_1) \sin(u_1 - z_{i_1}) \sin(u_1 + z_{i_1})}, \quad \text{for } b = 1. \quad (\text{B.5})
\end{aligned}$$

Some remarks are in order. The summation in (B.5) is not over the full set of values, namely, $i_1 \neq i_2 \cdots i_M$, (c.f. [30]). Also there are second order poles (see (B.6) below), which make the calculation tedious.

Taking advantage of the analytical properties of the trigonometric function and keeping in mind that $\{v_i^{(1)}\}$ are solutions of the first set of Bethe ansatz equations, we can calculate the sum in (B.5) to get

$$\begin{aligned}
&\sum_{i_1 \neq i_2, \dots, i_M}^M \frac{\prod_{\alpha=1}^M \sin(v_\alpha^{(1)} + z_{i_1}) \sin(v_\alpha^{(1)} - z_{i_1})}{\prod_{k>1}^M \sin(z_{i_1} - z_{i_k}) \sin(z_{i_k} + z_{i_1})} \\
&\times \frac{\sin(\lambda_1 + \xi - z_{i_1}) \sin(\lambda_2 + \xi + z_{i_1}) \sin(2u_1)}{\sin(\lambda_1 + \xi + u_1) \sin(\lambda_2 + \xi + u_1) \sin(u_1 - z_{i_1}) \sin(u_1 + z_{i_1})} \\
&\times \frac{\sin(\lambda_1 + \xi + z_{i_1}) \sin(\lambda_2 + \xi - z_{i_1}) \sin(2v_a^{(1)})}{\sin(\lambda_1 + \xi - v_a^{(1)}) \sin(\lambda_2 + \xi - v_a^{(1)}) \sin^2(v_a^{(1)} - z_{i_1}) \sin^2(v_a^{(1)} + z_{i_1})} \\
&= \frac{\widetilde{\mathcal{N}}_{a1}(u_1, \{v_\alpha^{(1)}\}; \{z_{i_1}\})}{\prod_{k=2}^M \sin(u_1 - z_{i_k}) \sin(u_1 + z_{i_k})} + \sum_{b=2}^n \alpha_b \mathcal{N}_{ab}^{(1)}(u_1, \{v_\alpha^{(1)}\}; \{z_{i_1}\}), \quad (\text{B.6})
\end{aligned}$$

with

$$\begin{aligned}
&\widetilde{\mathcal{N}}_{a1}(u_1, \{v_\alpha^{(1)}\}; \{z_i\}) \\
&= \frac{\sin(\lambda_1 + \xi - u_1) \sin(\lambda_2 + \xi - u_1) \prod_{\alpha=1}^M \sin(v_\alpha^{(1)} - u_1) \prod_{\alpha=1}^M \sin(v_\alpha^{(1)} + u_1) \sin(2v_a^{(1)}) \sin(2u_1)}{\sin(\lambda_1 + \xi - v_a^{(1)}) \sin(\lambda_2 + \xi - v_a^{(1)}) \sin^2(v_a^{(1)} - u_1) \sin^2(v_a^{(1)} + u_1)} \\
&\left[\sum_{k=1}^{2M} \frac{1}{\sin(u_1 - z_k) \sin(u_1 + z_k)} - \sum_{k \neq \alpha}^M \frac{2}{\sin(u_1 - v_k^{(1)}) \sin(u_1 + v_k^{(1)})} \right. \\
&\quad \left. + \frac{1 - \Delta}{\sin(\lambda_1 + \xi + u_1) \sin(\lambda_1 + \xi - u_1)} + \frac{1 + \Delta}{\sin(\lambda_2 + \xi + u_1) \sin(\lambda_2 + \xi - u_1)} \right], \quad (\text{B.7})
\end{aligned}$$

where α_b are coefficients which do not depend on a . Then only the first term of the r.h.s. of (B.6) has nonzero contribution to the determinant of $\widetilde{\mathcal{N}}$.

To show (B.6), we first prove the following relation:

$$\begin{aligned}
& \sum_{i_1 \neq i_2, \dots, i_M}^M \frac{\prod_{\alpha=1}^M \sin(v_\alpha^{(1)} + z_{i_1}) \sin(v_\alpha^{(1)} - z_{i_1})}{\prod_{k>1}^M \sin(z_{i_1} - z_{i_k}) \sin(z_{i_k} + z_{i_1})} \\
& \times \frac{\sin(\lambda_1 + \xi - z_{i_1}) \sin(\lambda_2 + \xi + z_{i_1}) \sin(2u_1)}{\sin(\lambda_1 + \xi + u_1) \sin(\lambda_2 + \xi + u_1) \sin(u_1 - z_{i_1}) \sin(u_1 + z_{i_1})} \\
& \times \frac{\sin(\lambda_1 + \xi + z_{i_1}) \sin(\lambda_2 + \xi - z_{i_1}) \sin(2v_a^{(1)})}{\sin(\lambda_1 + \xi - v_a^{(1)}) \sin(\lambda_2 + \xi - v_a^{(1)}) \sin^2(v_a^{(1)} - z_{i_1}) \sin^2(v_a^{(1)} + z_{i_1})} \\
& = \frac{\widetilde{\mathcal{N}}_{a1} \sin(u_1, \{v_\alpha^{(1)}\}; \{z_i\})}{\prod_{k=2}^M \sin(u_1 - z_{i_k}) \sin(u_1 + z_{i_k})} - \frac{1}{\sin(\lambda_1 + \xi + u_1) \sin(\lambda_2 + \xi + u_1)} \\
& \times \left[\sum_{b=2}^M \frac{1}{\sin^2(u_1 - z_{i_b})} \frac{R(z_{i_b})}{\prod_{k=2, k \neq b}^M \sin(z_{i_b} - z_{i_k}) \prod_{k=2}^M \sin(z_{i_b} + z_{i_k})} \right. \\
& + \sum_{b=2}^M \frac{1}{\sin(u_1 - z_{i_b})} \left\{ \frac{\mathcal{N}_{a, i_b}^{(1)}(u_1, \{v_\alpha^{(1)}\}; \{z_i\})}{\sin(\lambda_1 + \xi + z_{i_k}) \sin(\lambda_2 + \xi - z_{i_k})} \right. \\
& \left. \frac{\partial}{\partial u_1} \left(\frac{\prod_{\alpha=1}^M \sin(v_\alpha^{(1)} - u_1) \sin(v_\alpha^{(1)} + u_1) \sin(2u_1) \prod_{j=1}^2 \sin(\lambda_j + \xi + u_1) \sin(\lambda_j + \xi - u_1)}{\prod_{k=2, k \neq b}^M \sin(u_1 - z_{i_k}) \sin(u_1 + z_{i_k})} \right) \Big|_{u_1 = z_{i_b}} \right\} \\
& + \sum_{b=2}^M \frac{1}{\sin(u_1 - z_{i_b})} \frac{R(z_{i_b}) \sin(2z_{i_b})}{\prod_{k=2, k \neq b}^M \sin(z_{i_b} - z_{i_k}) \prod_{k=2}^M \sin(z_{i_b} + z_{i_k})} \\
& \times \left(\sum_{k=1}^{2M} \frac{1}{\sin(z_{i_b} - z_k) \sin(z_{i_b} + z_k)} - \sum_{k=1}^M \frac{2}{\sin(z_{i_b} - v_k^{(1)}) \sin(z_{i_b} + v_k^{(1)})} \right. \\
& \left. + \frac{1 - \Delta}{\sin(\lambda_1 + \xi + z_{i_b}) \sin(\lambda_1 + \xi - z_{i_b})} + \frac{1 + \Delta}{\sin(\lambda_2 + \xi + z_{i_b}) \sin(\lambda_2 + \xi - z_{i_b})} \right) \Big] \\
& + \text{trigonometric polynomials with poles of } \sin(u_1 + z_{i_k}), \tag{B.8}
\end{aligned}$$

where $R(z_{i_b})$ is

$$\begin{aligned}
R(z_{i_b}) &= \sin(\lambda_1 + \xi - z_{i_b}) \sin(\lambda_2 + \xi + z_{i_b}) \prod_{\alpha=1}^M \sin(v_\alpha^{(1)} - z_{i_b}) \prod_{\alpha=1}^M \sin(v_\alpha^{(1)} + z_{i_b}) \sin(2z_{i_b}) \\
& \times \frac{\sin(\lambda_1 + \xi + z_{i_b}) \sin(\lambda_2 + \xi - z_{i_b}) \sin(2v_a^{(1)})}{\sin(\lambda_1 + \xi - v_a^{(1)}) \sin(\lambda_2 + \xi - v_a^{(1)}) \sin^2(v_a^{(1)} - z_{i_b}) \sin^2(v_a^{(1)} + z_{i_b})} \\
& = \sin(\lambda_1 + \xi - z_{i_b}) \sin(\lambda_2 + \xi + z_{i_b}) \prod_{\alpha=1}^M \sin(v_\alpha^{(1)} - z_{i_b}) \prod_{\alpha=1}^M \sin(v_\alpha^{(1)} + z_{i_b}) \sin(2z_{i_b}) \\
& \times \mathcal{N}_{a, i_b}^{(1)}(u_1, \{v_\alpha^{(1)}\}; \{z_i\}). \tag{B.9}
\end{aligned}$$

Equation(B.8) can be proven by considering both sides as meromorphic functions of u_1 .

Then both sides of (B.8) have the same set of simple poles ²:

$$u_1 = \pm z_{i_1}, \mod(2\pi), \quad \text{where} \quad i_1 \neq i_2, \dots, i_M,$$

at which both have the same residues. Moreover, both sides tend 0 as $u_1 \rightarrow \infty$. We thus conclude that relation (B.8) holds. Note that that except the first one on the RHS of (B.8) all other terms can be expressed in the form of $\sum_b \alpha_b \mathcal{N}_{ab}^{(1)}(u_1, \{v_\alpha^{(1)}\}; \{z_i\})$. We thus obtain (B.6).

Repeating the above steps, by means of the recursive relations and changing the form of the determinant column by column, we obtain the final expression for $S(\{u_\alpha\}; \{v_i^{(1)}\})$.

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²We remark that the seemingly apparent poles on the RHS of (B.8), located at $\pm z_{i_2}, \dots, z_{i_M}$, are actually not poles.

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